§1.8. modules & finiteness conditions

R = ring.

- R-module = comm. gp M + scalar multiplication

  $(a+b)m \quad a(m+n) \quad (ab)\cdot m \quad 1\cdot m$

- example : 1) comm. gp

  2) vector space

- submodules e.g. $I \trianglelefteq R$, $W \subseteq V$ subvector sp. $H \trianglelefteq G$ subgp (com)

  finite generated R-module.

- several types of finiteness of an R-algebra

  ① (module-finite) finite R-alg. "+" + "R-linear"

  $\Downarrow$ e.g. # field / $\mathbb{Q}$. $k[x]/(f)$ $f \neq$ const.

  ② ring-finite finite generated R-alg. "+" "·"

  $\Downarrow$ $R[x_1 \cdots x_n]/I$

  "+" "·" "÷"

  ③ f.g. field extension. finite generated field extension

  i.e. $L = K(\alpha_1 \cdots \alpha_n)$

e.g. $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\pi]$, $\mathbb{Q}(\pi)$, $\mathbb{Q}[\pi, e]$, $\cdots$ $\underline{\mathbb{Q}(\sqrt{2})}$

$\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n$, $\mathbb{Q}[\alpha_1, \cdots, \alpha_n]$, $\mathbb{Q}(\alpha_1, \cdots, \alpha_n)$

§1.9. Integral elements.

Def $R \subset S$ subring. $v \in S$ integral over R, if $\exists$ monic poly. $F \in R[x]$ s.t. $F(v) = 0$.

If R and S are field and $v \in S$ is integral over R, then v is called algebraic over R

Prop $R \subset S$ subring of a domain. $v \in S$, TFAE:

1) v = integral over R
2) $R[v]$ = module-finite over R
3) $\exists$ subring $R[v] \subset R' \subset S$ s.t. R' is module-finite over R.

Pf: 1) $\Rightarrow$ 2). Suppose $v^n + a_1 v^{n-1} + \cdots + a_0 = 0 \Rightarrow R[v] = \sum_{i=0}^{n-1} R v^i$

2) $\Rightarrow$ 3). clear $R' := R[v]$

3) $\Rightarrow$ 1). $R' = \sum_{i=1}^{n} R w_i \Rightarrow e(w_1, \cdots, w_n) = (w_1 \cdots w_n) A$

$\Rightarrow (w_1, \cdots, w_n)(eI_n - A) = 0$

$\Rightarrow \det(eI_n - A) = 0. \Rightarrow \checkmark$

Cor The set $\overline{R}$ of elements of S that are integral over R is a subring of S containing R

Pf: $a, b \in \overline{R} \Rightarrow R[a,b]$ = f.g. R-mod.

$\Rightarrow a \pm b,\ a \cdot b \in \overline{R} \Rightarrow \overline{R}$ = subring

S is integral over $R \overset{def}{\Longleftrightarrow} \overline{R} = S$

(22) S is an algebraic extension of $R \overset{def}{\Longleftrightarrow} \begin{cases} S, R = \text{field} \\ \overline{R} = S \end{cases}$

§1.10 Field extension.

the field extension generated by a single element. → i.e. minimal subfield of L containing K and v.

Lem $K \subset L$ subfield. $L = \underline{K(v)}$. Then

1) $L \cong K(X)$, or
2) $L = K[v]$, where v is algebraic over K.

Pf: $\varphi : K[X] \to L$, $X \mapsto v$ $\Rightarrow$ $\ker\varphi = (F) \trianglelefteq K[X]$ (PID)

$\operatorname{im}\varphi$ = domain $\Rightarrow$ $\ker\varphi$ = prime

1° $F = 0 \Rightarrow L \cong K(X)$

2° $F \neq 0 \Rightarrow (F) \trianglelefteq K[X]$ maximal $\Rightarrow K[v]$ = field $\Rightarrow K[v] = K(v)$

(*) in proof of Nullstellensatz : $k = \bar{k}$, $L = k[v_1, \cdots v_n]$ = field. then $L = k$.

Prop (Zariski) L/K = field extension.

L = ring-finite over K (f.g. K-alg) $\Rightarrow$ L = module-finite over K (finite K-alg.)

Pf: Suppose $L = K[v_1, \cdots, v_n]$.

$n = 1$ (lem 1.10.1) ✓

$n-1$ ✓

$K_1 := K(v_1) \Rightarrow L = K_1(v_2, \cdots, v_n)$ = f.g. K_1-module.

Assume v_1 not algebraic over K (or, Problem 1.45(a) ⇒ ✓)

$\forall i \ \exists a_{ij} \in K$ s.t.

$$v_i^{n_i} + a_{i1} v_i^{n_i - 1} + \cdots = 0$$

$a \in K[v_1]$: multiple of all the denominators of a_{ij}.

$$\Rightarrow (av_i)^{n_i} + (a \cdot a_{i1})(av_1)^{n_i - 1} + \cdots = 0$$

$\Rightarrow av_2, \cdots av_n$ integral over $k[v_1]$.

$\Rightarrow \forall z \in L = k[v_1 \cdots v_n], \ \exists N > 0$ s.t. $a^N z$ integral over $K[v_1]$.

$\Rightarrow \forall z \in K(v_1), \ \exists N > 0$ s.t. $a^N z$ integral over $K[v_1]$. ↯

$\left(\text{e.g. } z = \frac{1}{v_1(a+1)} \Rightarrow \frac{a^m}{v_1(a+1)} \text{ not integral over } K[v_1] \ \forall m \geq 0. \right)$

Cor (Weak Nullstellensatz) : - - -

Pf - - -